

A Necessary and Sufficient Condition on the Weyl Manifolds Admitting a Semi Symmetric Non-Metric Connection to be S-Concircular

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November 14, 2014

Abstract

The object of this paper is to obtain the concircular curvature tensor of the semi symmetric non-metric connection on the Weyl manifold and to give a necessary and sufficient condition for a semi symmetric non-metric connection to be S-concircular.

Keywords: Weyl manifolds, semi symmetric non-metric connection, concircular curvature tensor, S-concircular connection.

Mathematical Subject Classification :53A40

1 Introduction

An n -dimensional manifold which has a symmetric connection ∇ and a conformal metric tensor g satisfying the compatibility condition

$$\nabla g = 2(T \otimes g)$$

where T is a 1-form is called a Weyl space which is denoted by $W_n(g, T)$, (see [1]). In local coordinates, the compatibility condition is given by

$$\nabla_k g_{ij} - 2g_{ij}T_k = 0 \quad (1.1)$$

where T_k is a complementary covariant vector field. Such a Weyl manifold will be denoted by $W_n(g_{ij}, T_k)$. If $T_k = 0$ or T_k is gradient, a Riemannian manifold is obtained.

In [1], under the transformation of the metric tensor g_{ij} in the form of

$$\widetilde{g}_{ij} = \lambda^2 g_{ij} \quad (1.2)$$

T_k changes by

$$\widetilde{T}_k = T_k + \partial_k (\ln \lambda),$$

where λ is a scalar function defined on W_n .

The coefficients Γ_{jk}^i of the symmetric connection ∇ on the Weyl manifold are defined by

$$\Gamma_{jk}^i = \{^i_{jk}\} - g^{im} (g_{mj} T_k + g_{mk} T_j - g_{jk} T_m) \quad (1.3)$$

where $\{^i_{jk}\}$'s are the coefficients of the Levi-Civita connection, (see [1]).

In [1], the coefficients Γ_{jk}^i and the curvature tensor R_{ijk}^h of the symmetric connection ∇ change by

$$\Gamma_{jk}^{i*} = \Gamma_{jk}^i + \delta_j^i P_k + \delta_k^i P_j - g_{jk} P^i \quad (1.4)$$

and

$$R_{ijk}^{h*} = R_{ijk}^h + 2\delta_i^h \nabla_{[j} P_{k]} + \delta_k^h P_{ij} - \delta_j^h P_{ik} + g_{ij} g^{hr} P_{rk} - g_{ik} g^{hr} P_{rj} \quad (1.5)$$

where $T_i - T_i^* = P_i$ and $P_{ij} = \nabla_j P_i - P_i P_j + \frac{1}{2} g_{ij} g^{kr} P_k P_r$ under conformal mapping $g_{ij}^* = g_{ij}$.

The conformal curvature tensor C_{ijk}^h and the concircular curvature tensor Z_{ijk}^h of the symmetric connection ∇ on the Weyl manifold are given by

$$\begin{aligned} C_{mijk} &= R_{mijk} - \frac{1}{n} g_{mi} R_{rjk}^r + \frac{1}{n-2} (g_{mj} R_{ik} - g_{mk} R_{ij} - g_{ij} R_{mk} + g_{ik} R_{mj}) \\ &\quad - \frac{1}{n(n-2)} (g_{mj} R_{rki}^r - g_{mk} R_{rji}^r - g_{ij} R_{rkm}^r + g_{ik} R_{rjm}^r) \\ &\quad - \frac{R}{(n-1)(n-2)} (g_{mj} g_{ik} - g_{mk} g_{ij}) \end{aligned} \quad (1.6)$$

and

$$Z_{mijk} = R_{mijk} - \frac{R}{n(n-1)} (g_{mk} g_{ij} - g_{mj} g_{ik}), \quad (1.7)$$

where R_{ijk}^h , R_{ij} and R denote the curvature tensor, Ricci tensor and scalar curvature tensor of ∇ , respectively, (see [2], [3]).

In [4], V.Murgescu defined the coefficients $\bar{\Gamma}_{jk}^i$ of a generalized connection $\bar{\nabla}$ on the Weyl manifold by

$$\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + a_{jkh} g^{hi} \quad (1.8)$$

where

$$a_{jkh} = g_{jr} \Omega_{kh}^r + g_{rk} \Omega_{jh}^r + g_{rh} \Omega_{jk}^r \quad (1.9)$$

and Γ_{jk}^i 's are the coefficients of the symmetric connection ∇ .

By choosing

$$\Omega_{jk}^i = \delta_j^i a_k - \delta_k^i a_j$$

in (1.9), the coefficients $\bar{\Gamma}_{jk}^i$'s of a semi symmetric non-metric connection $\bar{\nabla}$ on the Weyl manifold are obtained by

$$\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_k^i S_j - g_{jk} S^i \quad (1.10)$$

(see [5]). In (1.10), $S_i = -2a_i$ where a_i is an arbitrary covariant vector field.

The following results are also obtained in [5]:

The torsion tensor T_{jk}^i with respect to the semi symmetric connection $\bar{\nabla}$ is

$$T_{jk}^i = \delta_k^i S_j - \delta_j^i S_k \quad (1.11)$$

The curvature tensor $\bar{R}(X, Y)Z$ of the semi symmetric non-metric connection $\bar{\nabla}$ on the Weyl manifold is defined by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z$$

In local coordinates, above equation becomes

$$\bar{R}_{ijk}^h = \partial_j \bar{\Gamma}_{ik}^h - \partial_k \bar{\Gamma}_{ij}^h + \bar{\Gamma}_{rj}^h \bar{\Gamma}_{ik}^r - \bar{\Gamma}_{rk}^h \bar{\Gamma}_{ij}^r \quad (1.12)$$

By means of (1.10) and (1.12), the relation between the curvature tensors R_{ijk}^h and \bar{R}_{ijk}^h of ∇ and $\bar{\nabla}$, respectively, is obtained as

$$\bar{R}_{ijk}^h = R_{ijk}^h + \delta_k^h S_{ij} - \delta_j^h S_{ik} + g_{ij} g^{hr} S_{rk} - g_{ik} g^{hr} S_{rj} \quad (1.13)$$

where

$$S_{ij} = S_{i,j} - S_i S_j + \frac{1}{2} g_{ij} g^{kr} S_k S_r \quad (1.14)$$

and $S_{i,j}$ denotes the covariant derivative of S_i with respect to the symmetric connection ∇ .

Transvecting (1.13) by g_{mh} and contracting on the indices h and k in the same equation give

$$\bar{R}_{mijk} = R_{mijk} + g_{mk} S_{ij} - g_{mj} S_{ik} + g_{ij} S_{mk} - g_{ik} S_{mj} \quad (1.15)$$

and

$$\bar{R}_{ij} = R_{ij} + (n-2) S_{ij} + S g_{ij} \quad (1.16)$$

where $S = g^{mk} S_{mk}$, respectively.

The scalar curvatures R and \bar{R} of the connections ∇ and $\bar{\nabla}$, respectively, are related by

$$\bar{R} = R + 2(n-1) S \quad (1.17)$$

The curvature tensor of the semi-symmetric connection $\bar{\nabla}$ has the following properties:

- a $\bar{R}_{mijk} + \bar{R}_{mikj} = 0$,
- b $\bar{R}_{mijk} + \bar{R}_{imjk} = 4g_{mi}\nabla_{[k}T_{j]}$,
- c $\bar{R}_{rjk}^r = R_{rjk}^r = 2R_{[kj]} = 2n\nabla_{[k}T_{j]}$,
- d $\bar{R}_{mijk} + \bar{R}_{mjki} + \bar{R}_{mkij} = 2(g_{mi}\nabla_{[k}S_{j]} + g_{mj}\nabla_{[i}S_{k]} + g_{mk}\nabla_{[j}S_{i]})$.

The conformal curvature tensor \bar{C}_{mijk} of $\bar{\nabla}$ is given by

$$\begin{aligned}\bar{C}_{mijk} = & \bar{R}_{mijk} - \frac{1}{n}g_{mi}\bar{R}_{rjk}^r + \frac{1}{n-2}\{g_{mj}\bar{R}_{ik} - g_{mk}\bar{R}_{ij} - g_{ij}\bar{R}_{mk} + g_{ik}\bar{R}_{mj}\} \\ & - \frac{1}{n(n-2)}\{g_{mj}\bar{R}_{rki}^r - g_{mk}\bar{R}_{rji}^r - g_{ij}\bar{R}_{rkm}^r + g_{ik}\bar{R}_{rjm}^r\} \\ & - \frac{\bar{R}}{(n-1)(n-2)}\{g_{mj}g_{ik} - g_{mk}g_{ij}\}.\end{aligned}\quad (1.18)$$

The conformal curvature tensors C_{mijk} and \bar{C}_{mijk} of the connections ∇ and $\bar{\nabla}$ are related by

$$\bar{C}_{mijk} = C_{mijk}$$

The projective curvature tensor \bar{W}_{mijk} of $\bar{\nabla}$ is in the form of

$$\begin{aligned}\bar{W}_{mijk} = & \bar{R}_{mijk} + \frac{g_{mi}}{n+1}\{(\bar{R}_{jk} - \bar{R}_{kj}) + 2(n-1)\nabla_{[j}S_{k]}\} \\ & + \frac{1}{n^2-1}\{g_{mj}\bar{H}_{ik} - g_{mk}\bar{H}_{ij}\}\end{aligned}\quad (1.19)$$

where

$$\bar{H}_{ij} = n\bar{R}_{ij} + \bar{R}_{ji} + 2(n-1)\nabla_{[j}S_{i]}.$$

The projective curvature tensors W_{mijk} and \bar{W}_{mijk} of the connections ∇ and $\bar{\nabla}$ are related by the equation

$$\bar{W}_{mijk} = W_{mijk} + \frac{2}{n+1}g_{mi}\nabla_{[j}S_{k]} + \frac{1}{n^2-1}(g_{mk}K_{ij} - g_{mj}K_{ik}) + g_{ij}S_{mk} - g_{ik}S_{mj}$$

where

$$K_{ij} = nS_{ij} + S_{ji} + (n+1)Sg_{ij}.$$

2 Weyl manifolds admitting a semi symmetric non-metric connection under concircular mapping

Let $\sigma : (W_n, g_{ij}, T_k, S_k) \rightarrow (W_n^*, g_{ij}^*, T_k^*, S_k^*)$ be a conformal mapping given by $g_{ij}^* = g_{ij}$. In [5], according to this mapping, the coefficients $\bar{\Gamma}_{jk}^i$ and the curvature tensor \bar{R}_{ijk}^h of the semi symmetric connection $\bar{\nabla}$ change by :

$$\bar{\Gamma}_{jk}^{i*} = \bar{\Gamma}_{jk}^i + \delta_j^i P_k + \delta_k^i (P_j - Q_j) - g_{jk}(P^i - Q^i) \quad (2.1)$$

where

$$P_j = T_j - T_j^*, \quad Q_j = S_j - S_j^*$$

and

$$\begin{aligned} \overline{R}_{ijk}^{h*} &= \overline{R}_{ijk}^h + 2\delta_i^h (\nabla_{[j} P_{k]} + P_{[j} S_{k]}) + \delta_k^h W_{ij} - \delta_j^h W_{ik} + g_{ij} g^{hr} W_{rk} \\ &\quad - g_{ik} g^{hr} W_{rj} + 2g^{sr} P_s Q_r (\delta_j^h g_{ik} - \delta_k^h g_{ij}) \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} W_{ij} &= \underline{P}_{ij} - \underline{Q}_{ij} + 2P_{(i} Q_{j)}, \\ \underline{P}_{ij} &= P_{ij} - P_i S_j, \quad P_{ij} = \nabla_j P_i - P_i P_j + \frac{1}{2} g_{ij} g^{kr} P_k P_r, \\ \underline{Q}_{ij} &= Q_{ij} - Q_i S_j, \quad Q_{ij} = \nabla_j Q_i + Q_i Q_j - \frac{1}{2} g_{ij} g^{kr} Q_k Q_r. \end{aligned}$$

Since a conformal mapping which $W_{ij} = \phi g_{ij}$ changes a geodesic circle into a geodesic circle, it is called concircular mapping by means of [6].

Let σ be a concircular mapping, that is, \underline{P}_{ij} is symmetric. Then, (2.2) can be rewritten as follows:

$$\overline{R}_{ijk}^{h*} = \overline{R}_{ijk}^h + 2(\phi - g^{sr} P_s Q_r)(\delta_k^h g_{ij} - \delta_j^h g_{ik}) \quad (2.3)$$

By contracting on h and k in (2.3),

$$\overline{R}_{ij}^* = \overline{R}_{ij} + 2(n-1)(\phi - g^{sr} P_s Q_r) g_{ij} \quad (2.4)$$

Transvecting (2.4) by $g^{ij*} = g^{ij}$, yields

$$\overline{R}^* = \overline{R} + 2n(n-1)(\phi - g^{sr} P_s Q_r) \quad (2.5)$$

If the expression $2(\phi - g^{sr} P_s Q_r) = \frac{\overline{R}^* - \overline{R}}{n(n-1)}$ obtained from (2.5) is substituted in (2.3),

$$\overline{R}_{ijk}^{h*} = \overline{R}_{ijk}^h + \frac{\overline{R}^* - \overline{R}}{n(n-1)}(\delta_k^h g_{ij} - \delta_j^h g_{ik})$$

is arranged as

$$\overline{R}_{ijk}^{h*} - \frac{\overline{R}^*}{n(n-1)}(\delta_k^h g_{ij}^* - \delta_j^h g_{ik}^*) = \overline{R}_{ijk}^h - \frac{\overline{R}}{n(n-1)}(\delta_k^h g_{ij} - \delta_j^h g_{ik}).$$

If the *concircular curvature tensor* \overline{Z}_{ijk}^h is defined by

$$\overline{Z}_{ijk}^h = \overline{R}_{ijk}^h - \frac{\overline{R}}{n(n-1)}(\delta_k^h g_{ij} - \delta_j^h g_{ik}), \quad (2.6)$$

it is invariant under the concircular transformation, i.e.

$$\overline{Z}_{ijk}^{h*} = \overline{Z}_{ijk}^h.$$

First transvecting (2.6) by g_{mh} and then contracting on the indices h and k in (2.6), the equations

$$\overline{Z}_{mijk} = \overline{R}_{mijk} - \frac{\overline{R}}{n(n-1)}(g_{mk}g_{ij} - g_{mj}g_{ik}) \quad (2.7)$$

and

$$\overline{Z}_{ij} = \overline{R}_{ij} - \frac{\overline{R}}{n}g_{ij} \quad (2.8)$$

are obtained.

Lemma 1 *The concircular curvature tensor of the semi symmetric connection $\overline{\nabla}$ has the following properties:*

- a $\overline{Z}_{mijk} + \overline{Z}_{mikj} = 0$,
- b $\overline{Z}_{mijk} + \overline{Z}_{imjk} = 4g_{mi}\nabla_{[k}T_{j]}$,
- c $\overline{Z}_{rjk}^r = \overline{R}_{rjk}^r$,
- d $\overline{Z}_{mijk} + \overline{Z}_{mjki} + \overline{Z}_{mkij} = 0$.

The concircular curvature tensors Z_{ijk}^h and \overline{Z}_{ijk}^h of ∇ and $\overline{\nabla}$, respectively, are related by

$$\overline{Z}_{ijk}^h = Z_{ijk}^h + \delta_k^h S_{ij} - \delta_j^h S_{ik} + g_{ij}g^{mh}S_{mk} - g_{ik}g^{mh}S_{mj} - \frac{2}{n}S(\delta_k^h g_{ij} - \delta_j^h g_{ik}) \quad (2.9)$$

by substituting (1.7), (1.13) and (1.17) in (2.6).

Transvecting (2.9) by g_{mh} and contracting on h and k in the same equation give

$$\overline{Z}_{mijk} = Z_{mijk} + g_{mk}S_{ij} - g_{mj}S_{ik} + g_{ij}S_{mk} - g_{ik}S_{mj} - \frac{2}{n}S(g_{mk}g_{ij} - g_{mj}g_{ik}) \quad (2.10)$$

and

$$\overline{Z}_{ij} = Z_{ij} + (n-2)S_{ij} - \frac{(n-2)}{n}g_{ij}S \quad (2.11)$$

3 Semi symmetric non-metric S-Concircular Connection

In [7], Liang defined semi symmetric recurrent metric connection which is S-concircular on the Riemannian manifolds. In this paper, a semi symmetric non-metric S-concircular connection on the Weyl manifold is defined by as follows:

Definition 2 *If the semi symmetric non-metric connection $\bar{\nabla}$ satisfies the condition given by*

$$S_{ij} = \nabla_j S_i - S_i S_j + \frac{1}{2} g_{ij} g^{rs} S_r S_s = \beta g_{ij}$$

where β is a smooth function on the Weyl manifold, then it is called S-concircular.

Suppose that the concircular curvature tensors Z_{mijk} and \bar{Z}_{mijk} of symmetric and semi symmetric non-metric connections ∇ and $\bar{\nabla}$, respectively, be the same. Then

$$\bar{R}_{mijk} - \frac{\bar{R}}{n(n-1)}(g_{mk}g_{ij} - g_{mj}g_{ik}) = R_{mijk} - \frac{R}{n(n-1)}(g_{mk}g_{ij} - g_{mj}g_{ik}). \quad (3.1)$$

By using (1.15) in (3.1), we get

$$g_{mk}S_{ij} - g_{mj}S_{ik} + g_{ij}S_{mk} - g_{ik}S_{mj} = \frac{\bar{R} - R}{n(n-1)}(g_{mk}g_{ij} - g_{mj}g_{ik}) \quad (3.2)$$

By transvecting (3.2) by g^{mk} , it is obtained as

$$(n-2)S_{ij} + Sg_{ij} = \frac{\bar{R} - R}{n}g_{ij}.$$

By (1.17),

$$S_{ij} = \frac{\bar{R} - R}{2n(n-1)}g_{ij} \quad (3.3)$$

which states that $\bar{\nabla}$ is S-concircular.

Conversely, suppose that $\bar{\nabla}$ is S-concircular. By using $S_{ij} = \beta g_{ij}$, from Definition 2, in (2.10), it is obtained as

$$\bar{Z}_{mijk} = Z_{mijk}.$$

In the view of the above results, we can state the following theorem:

Theorem 3 *The necessary and sufficient condition for the semi symmetric non-metric connection $\bar{\nabla}$ to be S-concircular is that the concircular curvature tensors Z_{mijk} and \bar{Z}_{mijk} of the connections ∇ and $\bar{\nabla}$, respectively, coincide.*

References

- [1] Norden, A., Affinely connected spaces, GRMFL, Moscow (in Russian), 1976.
- [2] Miron, R., Mouvements conformes dans les espaces W_n , Tensor N.S., 1968, 19, 33-41.
- [3] Özdeğer, A., Şentürk, Z., Generalized Circles in Weyl Spaces and their conformal mapping, Publ. Math. Debrecen, 2002, 60, 1-2, 75-87.

- [4] Murgescu, V., Espaces de Weyl a torsion et leurs representations conformes, Ann. Sci. Univ. Timisoara, 1968, 221-228.
- [5] Unal, F.&Uysal, A., Weyl Manifolds with semi-symmetric connections, Mathematical and Computational Applications, 2005, Vol.10, No:3.
- [6] Yano, K., Concircular Geometry I. Concircular Transformations, Mathematical Institute, Tokyo Imperial University, 1940, 195-200.
- [7] Liang, Y., On semi symmetric recurrent metric S-concircular connections, Journal of Mathematical Study, 1994, 104-108.